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4D gravity localized in non $\mathbb{Z}_2$-symmetric thick branes

Nandinii Barbosa-Cendejas and Alfredo Herrera-Aguilar

Instituto de Física y Matemáticas, Universidad Michoacana de San Nicolás de Hidalgo
Edificio C-3, Cd. Universitaria, C.P. 58040 Morelia, Michoacán, México
E-mail: nandinii@ifm.umich.mx, herrera@ifm.umich.mx

Abstract: We present a comparative analysis of localization of 4D gravity on a non $\mathbb{Z}_2$-symmetric scalar thick brane in both a 5-dimensional riemannian space time and a pure geometric Weyl integrable manifold in which variations in the length of vectors during parallel transport are allowed and a geometric scalar field is involved in its formulation. This work was mainly motivated by the hypothesis which claims that Weyl geometries mimic quantum behaviour classically. We start by obtaining a classical 4-dimensional Poincaré invariant thick brane solution which does not respect $\mathbb{Z}_2$-symmetry along the (non-)compact extra dimension. This field configuration reproduces the $\mathbb{Z}_2$-symmetric solutions previously found in the literature, in both the Riemann and the Weyl frames, when the parameter $k_1 = 1$. The scalar energy density of our field configuration represents several series of thick branes with positive and negative energy densities centered at y_0. Thus, our field configurations can be compared with the standard Randall-Sundrum thin brane case. The only qualitative difference we have encountered when comparing both frames is that the scalar curvature of the riemannian manifold turns out to be singular for the found solution, whereas its weylian counterpart presents a regular behaviour. By studying the transverse traceless modes of the fluctuations of the classical backgrounds, we recast their equations into a Schödinger's equation form with a volcano potential of finite bottom (in both frames). By solving the Schödinger equation for the massless zero mode $m^2 = 0$ we obtain a single bound state which represents a stable 4-dimensional graviton in both frames. We also get a continuum gapless spectrum of KK states with positive $m^2 > 0$ that are suppressed at y_0, turning into continuum plane wave modes as y approaches spatial infinity. We show that for the considered solution to our setup, the potential is always bounded and cannot adopt the form of a well with infinite walls; thus, we do not get a discrete spectrum of KK states, and we conclude that the claim that weylian structures mimic, classically, quantum behaviour does not constitute a generic feature of these geometric manifolds.

Keywords: Brane Dynamics in Gauge Theories, Classical Theories of Gravity.

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Contents

1. Introduction

During last years it has shown an increasing interest in space times with large extra dimensions since in these models gravity propagates in all dimensions while matter is confined to a 4-dimensional submanifold (a 3-brane) with no contradiction with present time gravitational experiments [1]–[2]. In the framework of brane scenarios in 5-dimensional space time it has been shown a path towards the solution of some relevant problems of high-energy physics such as the cosmological constant, dark matter, non-locality and the mass hierarchy problem. In particular, it was discovered that in such brane scenarios 4-dimensional gravity can be realized consistently and we can live in $4 + 1$ non-compact dimensions in perfect compatibility with experimental gravity [3]–[5]. Since then, several generalizations of these scenarios have been constructed with the aid of thick branes [6]–[9]. In this paper we present a comparative study on the physics of a particular thick brane solution made out of scalar matter in two different manifolds: a riemannian space time and a purely geometric manifold endowed with Weyl structure. Thus, apart from the traditional riemannian treatment of the problem, where we have gravity coupled to a scalar field, we shall consider a non-riemannian generalization of 5-dimensional Kaluza-Klein theory by replacing the riemannian structure of the 5-dimensional manifold by a Weyl integrable geometry which allows for variations in the length of vectors during parallel transport and involves a geometric scalar field in its formulation. It has been shown that in certain cases weylian structures mimic classically quantum behaviour [10]. It is interesting to see whether or not we can obtain such a behaviour in our setup.

Thus, we begin by studying a 5-dimensional Weyl gravity model in which branes (thick domain walls) can be obtained naturally without introducing them by hand in the action of the theory. We further implement the conformal technique to obtain a classical solution that respects 4-dimensional Poincaré invariance and represents a localized function with no reflection $\mathbb{Z}_2$-symmetry and allows for both compact and non-compact manifolds in the extra dimension. By looking at the energy density of the scalar field of this solution we interpret the field configuration as a set of non $\mathbb{Z}_2$-symmetric thick branes. We investigate as well the behaviour of the curvature scalar and make an analysis of the fluctuations of the metric around the classical background solution in order to know whether 4-dimensional

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gravity can be described in our setup. It turns out that this is the case since the quantum mechanical problem with a potential well which vanishes asymptotically for the transverse traceless sector of the fluctuations of the metric yields a continuum spectrum of KK-states with a zero mode that corresponds to the normalizable, stable 4-dimensional graviton.

Let us consider a non-riemannian generalization of Kaluza-Klein theory, namely, the pure geometrical 5-dimensional action

$$S_5^W = \int_{M_5^W} \frac{d^5x\sqrt{|g|}}{16\pi G_5} e^{\frac{3}{2}\omega} \left[R + 3\tilde{\xi}\,(\nabla\omega)^2 + 6U(\omega)\right], \tag{1.1}$$

where M_5^W is a 5-dimensional Weyl integrable manifold determined by (g_{MN}, ω), g_{MN} is the metric $(M, N = 0, 1, 2, 3, 5)$ and ω is a scalar function. In such manifolds the Ricci tensor reads $R_{MN} = \Gamma^A_{MN,A} - \Gamma^A_{AM,N} + \Gamma^P_{MN}\Gamma^Q_{PQ} - \Gamma^P_{MQ}\Gamma^Q_{NP}$, where $\Gamma^C_{MN} = \{^{\ C}_{MN}\} - \frac{1}{2}\left(\omega_{,M}\delta^C_N + \omega_{,N}\delta^C_M - g_{MN}\omega^{,C}\right)$ are the affine connections and $\{^{\ C}_{MN}\}$ are the Christoffel symbols, the parameter $\tilde{\xi}$ is an arbitrary coupling constant, and $U(\omega)$ is a self-interaction potential for the scalar field. This action is of pure geometrical nature since the scalar field that couples to gravity is precisely the scalar function ω that enters in the definition of the affine connections of the Weyl manifold and, thus, cannot be discarded in principle from our consideration. Weyl integrable manifolds are invariant under the so-called Weyl rescalings

$$g_{MN} \to \Omega^{-2} g_{MN}\,, \qquad \omega \to \omega + \ln\Omega^2\,, \qquad \tilde{\xi} \to \frac{\tilde{\xi}}{(1 + \partial_\omega \ln\Omega^2)^2}, \tag{1.2}$$

where Ω^2 is a smooth function on M_5^W. In general, this invariance is broken by the self-interaction potential $U(\omega)$. From the relations (1.2) it follows that the transformation $U \to \Omega^2 U$ preserves such an invariance. Thus, the potential $U(w) = \lambda e^{\omega}$, where λ is a coupling constant, preserves the scale invariance of the Weyl action (1.1). When the Weyl invariance is broken, the scalar field is transformed into an observable degree of freedom which models the thick branes.

In order to find 4-dimensional Poincaré invariant solutions of the theory, we consider the following ansatz for the metric

$$ds_5^2 = e^{2A(y)}\eta_{mn}dx^m dx^n + dy^2\,, \tag{1.3}$$

where $e^{2A(y)}$ is the warp factor, $m, n = 0, 1, 2, 3$ and y labels the extra dimension.

By taking into account (1.3) we find the expressions for the Ricci tensor and the scalar curvature in the Weyl frame

$$\begin{aligned} R_{mn} &= -e^{2A}\left[A'' + 4(A')^2\right]\eta_{mn}\,, \qquad R_{55} = -4\left[A'' + (A')^2\right], \\ {}^5R &= -4\left[2A'' + 5(A')^2\right], \end{aligned} \tag{1.4}$$

where the comma denotes derivatives with respect to the fifth coordinate y. The 5-dimensional stress-energy tensor is given by

$$T_{MN} = \frac{1}{8\pi G_5}\left(R_{MN} - \frac{1}{2}g_{MN}R\right), \tag{1.5}$$

thus, its 4-dimensional and pure 5-dimensional components are given through the following expressions

$$T_{mn} = \frac{3}{8\pi G_5} e^{2A} \left[A'' + 2(A')^2 \right] \eta_{mn} \, , \qquad T_{55} = \frac{6(A')^2}{8\pi G_5} \, . \tag{1.6}$$

In order to find a solution to the setup defined by (1.1) and (1.3), we shall use the conformal technique: by means of a conformal transformation we go from the Weyl frame to the Riemann one, where the weylian affine connections become Christoffel symbols and the field equations are simpler, solve these equations and return to the Weyl frame to analyze the physics of the solution. By performing the conformal transformation $\widehat{g}_{MN} = e^{\omega} g_{MN}$ the action (1.1) is mapped into a riemannian manifold

$$S_5^R = \int_{M_5^R} \frac{d^5x\sqrt{|\widehat{g}|}}{16\pi G_5} \left[\widehat{R} + 3\xi \left(\widehat{\nabla}\omega \right)^2 + 6\widehat{U}(\omega) \right] , \tag{1.7}$$

where $\xi = \tilde{\xi} - 1$, $\widehat{U}(\omega) = e^{-\omega} U(\omega)$ and all the hatted magnitudes and operators refer to the Riemann frame. Thus, in this frame, we have a theory which describes 5-dimensional gravity coupled to a scalar field with a self-interaction potential. After this transformation, the line element is given by the following expression

$$\widehat{ds}_5^{\,2} = e^{2\sigma(y)} \eta_{nm} dx^n dx^m + e^{\omega(y)} dy^2 \, , \tag{1.8}$$

where $2\sigma = 2A + \omega$. Now the expressions for the Ricci tensor and the scalar curvature in the Riemann frame read

$$\begin{aligned} \widehat{R}_{mn} &= -e^{2A} \left[\sigma'' + 4(\sigma')^2 - \frac{1}{2}\omega'\sigma' \right] \eta_{mn}, \qquad \widehat{R}_{55} = -4 \left[\sigma'' + (\sigma')^2 - \frac{1}{2}\omega'\sigma' \right] , \\ \widehat{R} &= -4e^{-\omega} \left[2\sigma'' + 3(\sigma')^2 + 2\sigma' A' \right] . \end{aligned} \tag{1.9}$$

The 4-dimensional and pure 5-dimensional components of the stress-energy tensor are given by

$$\widehat{T}_{mn} = \frac{3}{8\pi G_5} e^{2A} \left[\sigma'' + 2(\sigma')^2 - \frac{1}{2}\omega'\sigma' \right] \eta_{mn} \, , \qquad \widehat{T}_{55} = \frac{6(\sigma')^2}{8\pi G_5} \, . \tag{1.10}$$

The solution. In order to find a solution for the model a new pair of variables $X \equiv \omega'$ and $Y \equiv 2A'$ are introduced. Thus, the corresponding to (1.7) field equations with the scaled line element (1.8) read [8]

$$\begin{aligned} X' + 2YX - \frac{3}{2}X^2 &= \frac{1}{\xi}\frac{d\widehat{U}}{d\omega} e^{-\omega}, \\ Y' + 2Y^2 - \frac{3}{2}XY &= \left(\frac{1}{\xi}\frac{d\widehat{U}}{d\omega} + 4\widehat{U} \right) e^{-\omega}. \end{aligned} \tag{1.11}$$

If we assume the condition $X = kY$, where k is an arbitrary constant this system of equations can be easily solved. Under this restriction, the potential $\widehat{U}$ must have the following form $\widehat{U} = \lambda e^{\frac{4k\xi}{1-k}\omega}$.

After imposing these conditions, the equation system (1.11) becomes

$$Y' + \frac{4-3k}{2}Y^2 = \frac{4\lambda}{1-k}e^{(\frac{4k\xi}{1-k}-1)\omega}\,. \qquad (1.12)$$

By setting $\xi = \frac{1-k}{4k}$, the field equation (1.12) simplifies further to

$$Y' + \frac{4-3k}{2}Y^2 = \frac{4\lambda}{1-k}\,. \qquad (1.13)$$

This choice accounts to having a self–interaction potential of the form $U = \lambda e^{2\omega}$ in the Weyl frame which, indeed, does not preserve the invariance of (1.1) under Weyl rescalings. By taking into account that $Y \equiv 2A'$ we get

$$A'' + (4-3k)(A')^2 - \frac{2\lambda}{1-k} = 0\,. \qquad (1.14)$$

By solving (1.14) we find the following solution

$$e^{2A(y)} = k_3\left(e^{ay} + k_1 e^{-ay}\right)^b\,, \qquad \omega = \ln\left[k_2\left(e^{ay} + k_1 e^{-ay}\right)^{kb}\right], \qquad (1.15)$$

where

$$a = \sqrt{\frac{4-3k}{1-k}2\lambda}\,, \qquad b = \frac{2}{4-3k}\,, \qquad (1.16)$$

and k_1, k_2 and k_3 are arbitrary constants.

This represents a solution which does not respect $\mathbb{Z}_2$-symmetry ($y \longrightarrow -y$) due to the presence of the constant parameter k_1. If we look at the particular case when $k_1 = 1$, $k_2 = 2^{-kb}$ and $k_3 = 2^{-b}$ we recover the $\mathbb{Z}_2$-symmetric solution previously found by [8] in the Weyl frame and by [6]–[7] in the Riemann one:

$$e^{2A(y)} = \left[\cosh(ay)\right]^b\,, \qquad \omega = bk\ln\left[\cosh(ay)\right]. \qquad (1.17)$$

In what follows we set $k_2 = k_3 = 1$ since these constants are not physically meaningful; they can be absorbed by a coordinate redefinition in the metric (1.8).

2. Physics of the system

Let us turn to study the physics of the found solution in both the Einstein and the Weyl frames. Since the conformal technique is used to generate the solution, we first analyze the behaviour of the warp factor, the energy density of the scalar field, the curvature scalar of the system as well as the metric fluctuations in the Riemann frame and after that we consider our solution and perform the same analysis in the Weyl frame. We finally compare both results.

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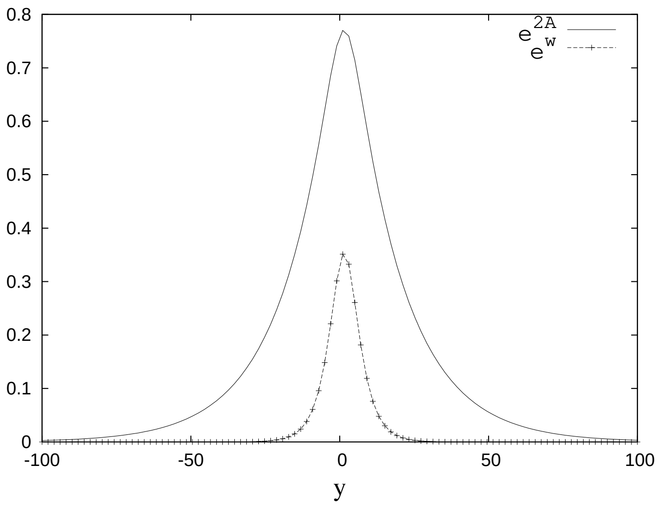

Figure 1: The shape of the warp factor e^{2A} and e^{ω} for $k > 4/3$, $\lambda > 0$, and $k_1 > 0$ is a smooth, localized and non $\mathbb{Z}_2$–symmetric function. In this particular figure we have set $k = 4$, $\lambda = 1/100$, and $k_1 = 2$ for both frames.

The Riemann frame. The warp factor and the scalar field are given by equations (1.15) and their shape depends on the values of the constants k_1, a, b. Thus, we have a family of solutions depending on the values of the parameters k_1, λ and k. We look for cases where the warp factor is a localized, smooth and well behaved function which in fact models the fifth dimension. By taking into account this fact, we have the following cases of interest:

(A) $k_1 > 0$, $\lambda > 0$ and $k > \frac{4}{3}$. In this case we have $a \in \Re$, $b < 0$, while the range of the fifth coordinate is $-\infty < y < \infty$, thus, we have a non-compact manifold in the extra dimension. It turns out that in this case, both the warp factor e^{2A} and e^{ω} are symmetric functions with respect to the point $y_0 = \ln k_1/(2a)$ and represent smooth, well behaved but non $\mathbb{Z}_2$-symmetric localized functions characterized by the width parameter $\Delta \sim 1/a$ (see figure 1).

It is easy to see that k_1 measures the $\mathbb{Z}_2$-asymmetry of the solution through a shift along the positive axis of the extra coordinate given by the value y_0. Since this constant appears multiplying an exponential function of y, its effect is quite small and, hence, the solution slightly deviates from the $\mathbb{Z}_2$-symmetric one. However, the physical implications of this fact are quite important, namely, the 5-dimensional space time is not restricted to be an orbifold geometry, allowing for a more general kind of dimensional reductions when going down to four dimensions.

The 4-dimensional and pure 5-dimensional components of the riemannian stress-energy tensor are given by the following expressions

$$\widehat{T}_{mn} = \frac{6a^2b(1+k)k_1}{8\pi G_5}(e^{ay} + k_1e^{-ay})^{b-2}\left[1 + \frac{b(k+2)}{8k_1}(e^{ay} - k_1e^{-ay})^2\right]\eta_{mn} \qquad (2.1)$$

and

$$\widehat{T}_{55} = \frac{3a^2b^2(1+k)^2}{16\pi G_5}\left(\frac{e^{ay} - k_1e^{-ay}}{e^{ay} + k_1e^{-ay}}\right)^2 . \qquad (2.2)$$

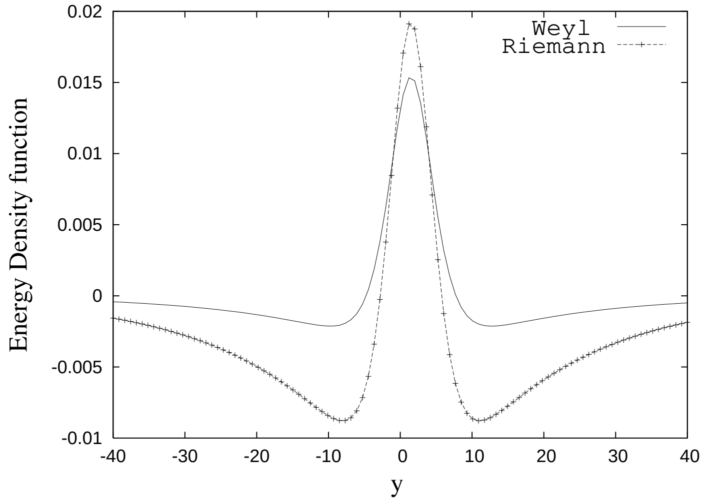

Figure 2: The shape of the energy density function with the same parameter values as in figure 1, but we have rescaled $\widehat{\mu}$ by $1/4$. Two thick branes with negative density energy are located around a brane with positive energy density centered at $y_0 = \ln k_1/(2a)$.

The energy density of the scalar matter is given by the null-null component of the stress-energy tensor

$$\widehat{\mu}(y) = \frac{-6a^2 b(1+k)k_1}{8\pi G_5}(e^{ay} + k_1 e^{-ay})^{b-2}\left[1 + \frac{b(k+2)}{8k_1}(e^{ay} - k_1 e^{-ay})^2\right]. \qquad (2.3)$$

The shape of the energy density $\widehat{\mu}$ shows a positive maximum at y_0 and two negative minima around the maximum; the function vanishes asymptotically for $y = \pm\infty$. Thus, this configuration can be interpreted as a thick brane with positive energy density, where the scalar matter is confined, located between two thick branes with negative energy density (see figure 2). This thick brane configuration becomes $\mathbb{Z}_2$-symmetric in the particular case when $k_1 = 1$, resembling the thin brane Randall-Sundrum model [3].

The 5-dimensional curvature scalar

$$\widehat{R}_5 = -16a^2 b(k+1)k_1(e^{ay} + k_1 e^{-ay})^{-(kb+2)}\left[1 + \frac{b(5+3k)}{16k_1}(e^{ay} - k_1 e^{-ay})^2\right] \qquad (2.4)$$

is not bounded and we have a 5-dimensional manifold which is singular in the Riemann frame.

(B) $k_1 > 0$, $\lambda > 0$ and $1 < k < 4/3$. In this case we have $a \in \Im$, $b > 0$ and we must replace $a \to i\alpha$ in order to have a real warp factor. Moreover, the only possible choice for the parameter k_1 is $k_1 = 1$ (otherwise the solution becomes complex), and we get a $\mathbb{Z}_2$-symmetric function

$$e^{2A(y)} = \cos^b(\alpha y)\,. \qquad (2.5)$$

Thus, this represents a manifold which is periodic in the extra dimension, so $-\pi \leq \alpha y \leq \pi$, and we have the same compact case that was obtained in [8].

The physically relevant case $\lambda < 0$, $k < 1$, $k_1 > 0$ coincides with case (B), while the cases $\lambda > 0$, $1 < k < 4/3$, $k_1 < 0$ and $\lambda < 0$, $k < 1$, $k_1 < 0$ yield y-periodic warp factors proportional to $\sin^b(ay)$ for positive b. These cases also correspond in essence to the case (B) shifted along the fifth dimension. All other possible combinations of the parameters do not yield physically relevant cases since for them, the warp factor infinitely increases as y approaches certain values and thus, it does not constitute a localized function.

Metric fluctuations. In order to see whether we can describe 4-dimensional gravity within our setup, we will examine fluctuations of the metric around a classical background solution. Thus, the perturbed metric (1.8) reads

$$\widehat{ds}^2 = e^{2\sigma(y)}[\eta_{mn} + \widehat{h}_{mn}(x,y)]dx^m dx^n + e^{\omega(y)} dy^2 \,. \tag{2.6}$$

In general, it is not possible to avoid considering fluctuations of the scalar field while treating fluctuations of the background metric. However, there exists a sector of the metric that decouples from the scalar field and the metric fluctuations can be suited analytically [6].

Following this line, we introduce a new r coordinate $dr = e^{-A}dy$ which yields a conformally flat metric

$$\widehat{ds}^2 = e^{2\sigma(y)} \left\{[\eta_{mn} + \widehat{h}_{mn}(x,y)]dx^m dx^n + dr^2\right\} \tag{2.7}$$

and the corresponding wave equation for the transverse traceless modes $\widehat{h}^T_{mn}$ that decouple from the scalar field

$$\left(\partial_r^2 + 3\sigma'\partial_r + \Box^\eta\right)\widehat{h}^T_{mn} = 0 \,, \tag{2.8}$$

where $\Box^\eta$ is the (flat) Minkowski wave operator. In [6] it has been shown that this equation supports a naturally massless and normalizable 4-dimensional graviton, since it is massless, it must obey $\Box^\eta \widehat{h}^T_{mn} = 0$, and we can easily see that such a solution is given by the following ansatz

$$\widehat{h}^T_{mn} = C_{mn} e^{imx} \,, \tag{2.9}$$

where the C_{mn} are constants and $m^2 = 0$.

For studying the massive KK-excitations it is useful to convert equation (2.8) into a Schrodinger's equation form. In order to achieve this aim we make the ansatz $\widehat{h}^T_{mn} = e^{imx}e^{-3\sigma/2}\Psi_{mn}(r)$ and the equations simplify further to

$$[\partial_r^2 - \widehat{V}(r) + m^2]\Psi = 0 \,, \tag{2.10}$$

where we have dropped the subscripts in Ψ, m is the mass of the KK-excitation and the potential $\widehat{V}(r)$ reads

$$\widehat{V}(r) = \frac{3}{2}\partial_r^2\sigma + \frac{9}{4}(\partial_r\sigma)^2 \,. \tag{2.11}$$

The shape of the potential $\widehat{V}(r)$ for the case (A) of the solution (1.15) is given by the following expression:

$$\widehat{V}(r(y)) = 3a^2 b(k+1)k_1(e^{ay} + k_1 e^{-ay})^{b-2}\left[1 + \frac{(3k+5)b}{16k_1}(e^{ay} - k_1 e^{-ay})^2\right] . \tag{2.12}$$

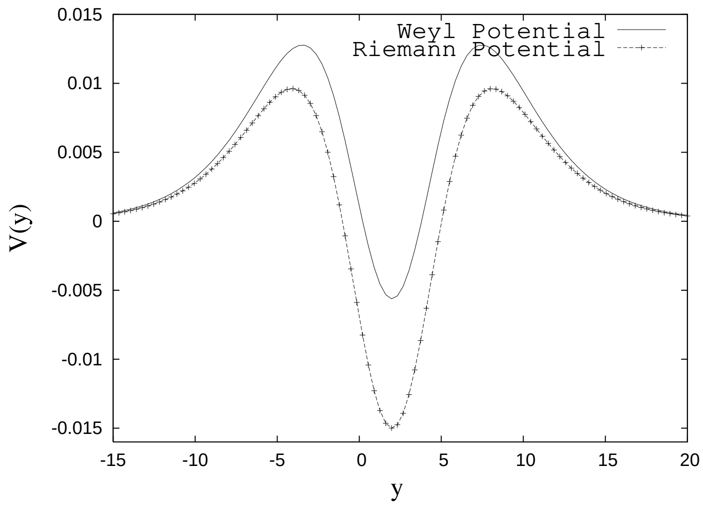

Figure 3: The shape of the potential $V(y)$. For this figure we have set $k = 5/3$, $\lambda = 1/100$, and $k_1 = 2$ in both frames.

By looking at figure 3 we see that this potential represents a well with a finite negative minimum at a certain value y_0, which is located between two positive maxima (potential barriers) and then vanishes as $y = \pm\infty$ (a volcano potential with finite bottom).

In the particular case $k = 5/3$ (hence $b = -2$), the coordinate transformation can be successfully inverted and we get $\sigma(r) = -4\ln[(a^2r^2 + 4k_1)]/3$ which yields a potential of the form

$$\widehat{V}(r) = \frac{4a^2\left(5a^2r^2 - 4k_1\right)}{\left(a^2r^2 + 4k_1\right)^2}\,. \tag{2.13}$$

The spectrum of eigenvalues m^2 parameterizes the spectrum of graviton masses that a 4-dimensional observer located at y_0 sees. It turns out that for the zero mode $m^2 = 0$, the Schrödinger equation (2.10) can be solved. The only normalizable eigenfunction reads $\Psi_0 = c\left(a^2r^2 + 4k_1\right)^{-2}$, where c is a normalization constant. Since this function has no zeros, it represents the lowest energy eigenfunction of the Schödinger equation (2.10), allowing for the existence of a 4-dimensional graviton with no instabilities from transverse traceless modes with $m^2 < 0$; moreover, since the potential vanishes asymptotically, this is the only gravitational bound state. In addition there exists a tower of higher KK modes with positive $m^2 > 0$. Thus, we have obtained non $\mathbb{Z}_2$-symmetric thick brane generalization of the Randall-Sundrum model in which the 4-dimensional effective theory possesses an energy spectrum quite similar to the spectrum of the thin wall case. Similar results were obtained in [7], but with just one free parameter.

The Weyl frame. Once we have obtained the solution in the Riemann frame by means of the conformal technique, we come back to the Weyl frame to study its physics. In this frame, for the non-compact case A) $\lambda > 0$, $k > \frac{4}{3}$ and $k_1 > 0$, the 4-dimensional and pure 5-dimensional components of the stress-energy tensor are given through the following

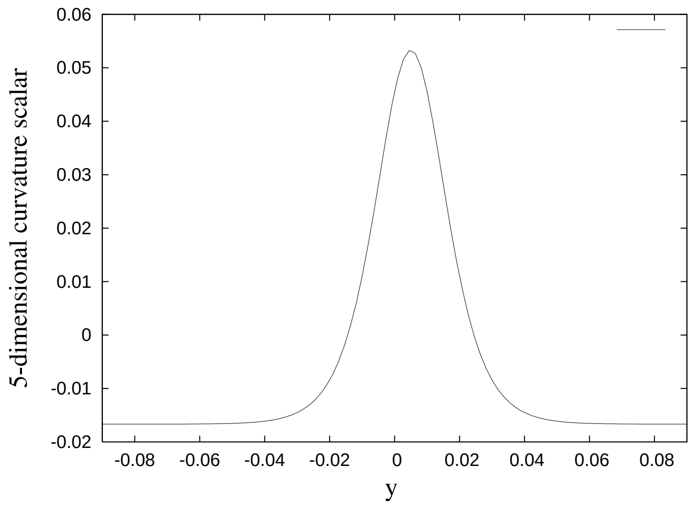

Figure 4: The functional form of the curvature scalar R_5 for $k > 4/3$ in the Weyl frame. This magnitude is singular in the Riemann frame.

expressions

$$T_{mn} = \frac{6a^2bk_1}{8\pi G_5}(e^{ay}+k_1e^{-ay})^{b-2}\left[1+\frac{b}{4k_1}(e^{ay}-k_1e^{-ay})^2\right]\eta_{mn} \tag{2.14}$$

and

$$T_{55} = \frac{3a^2b^2}{16\pi G_5}\left(\frac{e^{ay}-k_1e^{-ay}}{e^{ay}+k_1e^{-ay}}\right)^2 . \tag{2.15}$$

Thus, the energy density of the scalar matter is given by

$$\mu(y) = \frac{-6a^2bk_1}{8\pi G_5}(e^{ay}+k_1e^{-ay})^{b-2}\left[1+\frac{b}{4k_1}(e^{ay}-k_1e^{-ay})^2\right]. \tag{2.16}$$

This function also shows two negative minima and a positive maximum between them and, finally, it vanishes asymptotically. Thus, it presents the same behaviour as the energy density in the Riemann frame. It is displayed in figure 3, together with the same riemannian quantity $\widehat{\mu}$ to facilitate their comparison.

The 5-dimensional curvature scalar reads

$$R_5 = \frac{-16a^2bk_1}{(e^{ay}+k_1e^{-ay})^2}\left[1+\frac{5b}{16k_1}(e^{ay}-k_1e^{-ay})^2\right]. \tag{2.17}$$

From figure 4 we can see that this quantity is always bounded, thus, in contrast with the singular manifold that arises in the Riemann frame, we have a 5-dimensional manifold that is regular in the Weyl frame.[1]

[1]We thank the authors of [8] for clarifying the discrepancy between some of their expressions and ours in the limit $k_1 = 1$; they kindly communicated us that they made a systematic mistake when computing the quantities corresponding to (2.14)–(2.17) in their work.

Metric fluctuations. Now we want to study the metric fluctuations h_{mn} in the Weyl frame. In order to achieve this goal, we consider perturbations of the metric (1.3)

$$ds_5^2 = e^{2A(y)}[\eta_{mn} + h_{mn}(x,y)]dx^m dx^n + dy^2 \tag{2.18}$$

and perform the coordinate transformation $dw = e^{-A}dy$ which leads to a conformally flat metric and to the following wave equation for the transverse traceless modes h^T_{mn} of the metric fluctuations

$$(\partial_w^2 + 3A'\partial_w + \Box^\eta)h^T_{mn} = 0\,. \tag{2.19}$$

It is easy to see that this equation supports as well a massless and normalizable 4-dimensional graviton in the Weyl frame given by $h^T_{mn} = K_{mn}e^{imx}$, where K_{mn} are constants and $m^2 = 0$.

In order to recast equation (2.19) into a Schrödinger's equation form we adopt the ansatz $h^T_{mn} = e^{imx}e^{-3A/2}\psi_{mn}(w)$, and equation (2.19) simplifies further to

$$[\partial_w^2 - V(w) + m^2]\psi = 0\,, \tag{2.20}$$

where we have dropped the subscripts in ψ, m is the mass of the KK-excitation, and the potential reads

$$V(w) = \frac{3}{2}\partial_w^2 A + \frac{9}{4}(\partial_w A)^2\,. \tag{2.21}$$

The shape of the potential $V(w)$ in this set-up is given by the following expression

$$V(w(y)) = 3a^2bk_1(e^{ay}+k_1e^{-ay})^{b-2}\left[1+\frac{5b}{16k_1}(e^{ay}-k_1e^{-ay})^2\right] \tag{2.22}$$

and its behaviour is displayed in figure 3. From (2.22) we observe that it possesses a similar structure in comparison to its riemannian analogue; this is also confirmed by the shape of the potential in figure 3.

Let us consider the case $k = 5/3$ ($b = -2$) in which the coordinate transformation $dw = e^{-A}dy$ can be inverted. The function A reads $A(w) = -\ln\left(a^2w^2 + 4k_1\right)/2$ and yields a potential of the form

$$V(r) = \frac{3a^2\left(5a^2w^2 - 8k_1\right)}{4\left(a^2w^2 + 4k_1\right)^2}. \tag{2.23}$$

Once again, the Schrödinger equation (2.20) can be solved for the massless zero mode $m^2 = 0$, leading to the only normalizable eigenfunction $\psi_0 = q\left(a^2w^2 + 4k_1\right)^{3/4}$, where q is a normalization constant, which represents the 4-dimensional graviton with no instabilities from transverse traceless modes with $m^2 < 0$. Thus, the spectrum of graviton masses that a 4-dimensional observer located at y_0 sees in the Weyl frame is qualitatively the same as that of the Riemann frame and we have as well a continuum spectrum of KK states with positive $m^2 > 0$.

This fact contradicts the result obtained in [8] where a discrete spectrum of massive KK modes was obtained. It seems to us that this is a consequence of the form of their potential (see equation (26) in that work), where the power of the hyperbolic cosine function

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is the opposite to ours. Let us clarify this point. By setting $k_1 = 1$ in our solution we get the solution obtained by Arias et al. Further, by substituting the expression for $A(y)$ in the formula for the potential (2.21), we get an expression proportional to the warp factor $\cosh^b(ay)$ multiplied by $[1 + \beta \tanh^2(ay)]$, where β is a constant. One can see that both factors are bounded magnitudes for any real a and $b < 0$ (this is precisely the particular non-compact case we are considering). Thus, there is no way to obtain a potential well with infinite rigid walls and, therefore, the spectrum of KK states turns out to be continuum and gapless. Thus, we conclude that for the setup considered in the present work, Weyl geometries do not mimic classically quantum behaviour.

3. Conclusions and discussion

Along with the study of the physics of our thick brane solution in a 5-dimensional riemannian space time we have also considered it in a geometric generalization of Kaluza-Klein theory obtained by replacing the riemannian structure of the 5-dimensional manifold with a pure geometric Weyl integrable manifold in which variations in the length of vectors during parallel transport are allowed and a geometric scalar field is involved in its formulation. This was motivated by the hypothesis that Weyl geometries mimic quantum behaviour classically [8]. However, as a result of our investigation, we have shown that this is not a generic feature of such manifolds.

By means of the conformal technique we obtained a classical 4-dimensional Poincaré invariant solution which represents a well behaved localized function (a thick brane characterized by the width parameter $\Delta \sim 1/a$) which does not respect $\mathbb{Z}_2$-symmetry along the extra dimension. Thus, we have obtained a thick brane generalization of the Randall-Sundrum model. This field configuration does not restrict the 5-dimensional space time to be an orbifold geometry, allowing for arbitrary positive and negative values along the fifth coordinate y. This fact can be exploited when addressing difficult cosmological issues such as the cosmological constant problem, black hole physics [5] and holographic ideas, where there exists a correspondence between location in the fifth dimension and mass scale [11], since one has a better chance of solving these problems when all possible mass scales (all possible distances along the fifth dimension) are available. When we set the parameter $k_1 = 1$, our solution reproduces the $\mathbb{Z}_2$-symmetric solutions previously found in the literature in both the Riemann and the Weyl frames.

By looking at the scalar energy density μ of our field configuration, we see that it shows a thick brane with positive energy density centered at y_0 and accompanied by one thick brane with negative energy density at each side. The fact that we have both negative and positive values for the scalar energy density can be compared with the standard Randall-Sundrum thin brane case. The generic behaviour of this physical quantity is the same in both the Riemann and the Weyl frame.

The scalar curvature of the riemannian manifold turns out to be singular for the found solution, whereas the corresponding quantity in the Weyl integrable geometry presents a regular behaviour along the whole fifth dimension. It is interesting to check whether this is a generic property of the considered models.

By studying the behaviour of the transverse traceless modes of the fluctuations of the metric we recast their equations into a one dimensional Schödinger equation with a quantum mechanical potential that represents a finite negative well with a finite positive barrier at each side which then vanishes asymptotically (a volcano potential with finite bottom). We solve the Schödinger equation for the massless zero mode $m^2 = 0$ in both frames obtaining a single bound state which represents the lowest energy eigenfunction of the Schödinger equation, allowing for the existence of a 4-dimensional graviton with no instabilities from transverse traceless modes with $m^2 < 0$. We also get a huge tower of higher KK states with positive $m^2 > 0$ that are suppressed at y_0, turning into continuum plane wave modes as y approaches spatial infinity [5]–[6]. Since all values of m^2 are allowed, the spectrum turns out to be continuum and gapless.

We prove that for our setup, the analog quantum mechanical potential does not possess the shape of a well with infinite rigid walls. Due to this fact, one cannot get a discrete (quantized) spectrum of KK states when analyzing fluctuations of the metric. Thus, we conclude that the claim that weylian structures mimic, classically, quantum behaviour [8, 10] does not constitute a generic feature of these geometric manifolds.

Due to the dependence of the quantum mechanics potential on the warp factor functions, different warp factors will lead to distinct potentials and these, in turn, will alter the Schrödinger equation. When solving it, the new solutions will describe wave functions with different physics. Thus, it is important to consider other solutions to the original setup (both in Einstein and Weyl frames) and analyze the physics they yield. Another interesting issue that is currently under investigation is the solution of the mass hierarchy problem in 5D space times involving thick brane configurations.

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